

DSP: Lec 7

Quiz

① using z.T: find $y(n)$ in closed form for the difference eqn:-

$$y(n) = x(n-2) - x(n) + 0.5 y(n-1)$$

where $x(n) = \begin{cases} 0 & \text{at } n \text{ even} \\ 1 & \text{at } n \text{ odd} \end{cases}$

② $x(n) = \{-1, 1, 0, \frac{1}{2}, -\frac{1}{2}\}$

Find $* x((-n))_5 \quad * \quad x((n-2))_5$
 $* \quad x((2-n))_5$

② $\overset{\text{Sol}}{x((-n))_5 = \{-1, -\frac{1}{2}, \frac{1}{2}, 0, 1\}}$

$$x((n-2))_5 = \{\frac{1}{2}, -\frac{1}{2}, -1, 1, 0\}$$

$$x((2-n))_5 = \{0, 1, -1, \frac{1}{2}, -\frac{1}{2}\}$$

$$\boxed{1} \quad Y(z) = z^{-2} X(z) + X(z) + 0.5 z^{-1} Y(z)$$

$$Y(z) - 0.5 z^{-1} Y(z) = z^{-2} X(z) + X(z)$$

$$Y(z) = \frac{z^{-2} + 1}{1 - 0.5 z^{-1}} X(z)$$

$$X(z) = \sum_{n=0}^{\infty} x(n) z^{-n}$$

if $n \rightarrow \text{odd}$

$$X(z) = \sum_{n=0}^{\infty} z^{-n} = z^{-1} + z^{-2} + z^{-3} + \dots$$

$$= \frac{z^{-1}}{1 - z^{-2}}$$

$$Y(z) = \frac{z^{-2} + 1}{1 - 0.5 z^{-1}} * \frac{z^{-1}}{1 - z^{-2}} = \frac{z^{-2} + 1}{1 - 0.5 z^{-1}} * \frac{z^{-1}}{-(z^{-2} - 1)}$$

$$= \frac{-z^{-1}}{1 - 0.5 z^{-1}} = \frac{-1 * z * z^{-1}}{z - 0.5}$$

$$y(n) = -(0.5)^{n-1} u(n-1)$$

* Circular Convolution

note

Linear Convolution

$$y(n) = x_1(n) * x_2(n) = \sum_{k=-\infty}^{\infty} x_1(k) \cdot x_2(n-k)$$

$$= \sum_{k=-\infty}^{\infty} x_1(n-k) \cdot x_2(k)$$

* If the sequences (Discrete time sequences) are periodic, then the convolution is circular.

$$y(n) = x_1(n) \textcircled{N} x_2(n)$$

where $x_1(n)$, $x_2(n)$ are two periodic sequences every N samples.

$\textcircled{N} \Rightarrow$ the symbol of circular convolution.

$$y(n) = x_1(n) \textcircled{N} x_2(n)$$

$$= \sum_{m=0}^{N-1} x_1(m) \cdot x_2(n-m)$$

$$= \sum_{m=0}^{N-1} x_1(n-m) \cdot x_2(m)$$

$$= \text{IDFT} \{ X_1(K) \cdot X_2(K) \}$$

$X_1(K) \rightarrow \text{DFT}$ for the sequence $x_1(n)$

$X_2(K) \rightarrow$ " " " " " $x_2(n)$

EX Compute the circular convolution for:-

$x_1(n) = \{2, 1, 2, 1\}$ and $x_2(n) = \{1, 2, 3, 4\}$

For $N=4$ → Circular Convolution

or find $x_1(n) \textcircled{N} x_2(n)$

$$y(n) = x_1(n) \textcircled{4} x_2(n) = \sum_{m=0}^3 x_1(m) \cdot x_2((n-m))_4$$

m	0	1	2	3
$X_1(m)$	2	1	2	1
$X_2(m)$	1	2	3	4
$n=0$ $X_2((-m))_4$	1	4	3	2
$n=1$ $X_2((1-m))_4$	2	1	4	3
$n=2$ $X_2((2-m))_4$	3	2	1	4
$n=3$ $X_2((3-m))_4$	4	3	2	1

قاعدة التحويل
بـ n اني تعرف أجيب

$$\Rightarrow y(0)$$

$$\Rightarrow y(1)$$

$$\Rightarrow y(2)$$

$$\Rightarrow y(3)$$

$$y(0) = 1 \times 2 + 1 \times 4 + 3 \times 2 + 2 \times 1 = 14$$

$$y(1) = X_2((1-m))_4 \times X_1(m) = 2 \times 2 + 1 \times 1 + 2 \times 4 + 1 \times 3 = 16$$

$$y(2) = X_2((2-m))_4 \times X_1(m) = 14$$

$$y(3) = X_2((3-m))_4 \times X_1(m) = 16$$

$$y(n) = \{14, 16, 14, 16\}$$

ممکنہ طریقہ ثانیہ

$$y(n) = x_1(n) \textcircled{4} x_2(n)$$

$$x_2((l-m)) \rightarrow \begin{pmatrix} 1 & 4 & 3 & 2 \\ 2 & 1 & 4 & 3 \\ 3 & 2 & 1 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ 2 \\ 1 \end{pmatrix}$$

$N \times N$ $N \times 1$
 4×4 4×1

$$= \begin{pmatrix} 14 \\ 16 \\ 14 \\ 16 \end{pmatrix}$$

$$\Rightarrow y(n) = \{14, 16, 14, 16\}$$

$$\underline{\text{or}} \quad y(n) = \sum_{m=0}^3 x_2(m) \cdot x_1(n-m)$$

$$= \begin{pmatrix} 2 & 1 & 2 & 1 \\ 1 & 2 & 1 & 2 \\ 2 & 1 & 2 & 1 \\ 1 & 2 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 14 \\ 16 \\ 14 \\ 16 \end{pmatrix}$$

another solution

$$y(n) = x_1(n) \quad (4) \quad x_2(n) = \text{IDFT} (X_1(K) \cdot X_2(K))$$

□ Compute DFT for $x_1(n)$ & $x_2(n)$

$$X_1(K) = \begin{pmatrix} 6 \\ 0 \\ 2 \\ 0 \end{pmatrix}, \quad X_2(K) = \begin{pmatrix} 10 \\ -2+2j \\ -2 \\ -2-2j \end{pmatrix}$$

$$X_1(K) \cdot X_2(K) = \begin{pmatrix} 6 \\ 0 \\ 2 \\ 0 \end{pmatrix} \begin{pmatrix} 10 \\ -2+2j \\ -2 \\ -2-2j \end{pmatrix} = \begin{pmatrix} 60 \\ 0 \\ -4 \\ 0 \end{pmatrix}$$

$$y(n) = \text{IDFT} (X_1(K) \cdot X_2(K))$$

$$= \begin{pmatrix} 14 \\ 16 \\ 14 \\ 16 \end{pmatrix}$$

Prove the Following Properties

$$\boxed{1} \quad W_N^{K+N} = W_N^K$$

$$\boxed{2} \quad W_N^{K+\frac{N}{2}} = -W_N^K$$

$$\boxed{3} \quad W_N^2 = W_{N/2}$$

$$W_N = e^{-j \frac{2\pi}{N}} \quad \text{"twiddle factor"}$$

$$\boxed{1} \quad W_N^{K+N} = W_N^K \cdot W_N^N$$

$$W_N^N = e^{-j \frac{2\pi}{N} \times N} = e^{-j2\pi} = 1$$

$$\therefore W_N^{K+N} = W_N^K \quad \neq$$

$$\boxed{2} \quad W_N^{K+\frac{N}{2}} = -W_N^K$$

$$W_N^{K+\frac{N}{2}} = W_N^K \cdot W_N^{N/2}$$

$$W_N^{N/2} = e^{-j \frac{2\pi}{N} \times \frac{N}{2}} = e^{-j\pi} = \cos(180^\circ) - j \sin(180^\circ) = -1$$

$$\therefore W_N^{K+\frac{N}{2}} = -W_N^K$$

$$\boxed{3} \quad W_N^2 = W_{N/2}$$

$$W_{N/2} = e^{-j \frac{2\pi}{N/2}} = \left(e^{-j \frac{2\pi}{N}} \right)^2 \\ = W_N^2$$

* Fast Fourier Transform (FFT) Algorithms :-

→ are many algorithms used to reduce the complex computation for DFT

1] Radix-2 DIT FFT

DIT \Rightarrow Decimation (splitting) in time.

\Rightarrow For periodic sequence $x(n)$, the discrete Fourier transform is $X(k)$ which is periodic N samples.

$$\begin{array}{ccc} x(n) & \xrightarrow[N]{\text{DFT}} & X(k) \\ \downarrow & & \text{DFT} \\ \text{Discrete} & & \\ \text{time Sequence} & & \end{array}$$

$$X(K) = \sum_{n=0}^{N-1} x(n) W_N^{Kn}$$

الصورة القياسية.

Where: $W_N = e^{-j \frac{2\pi}{N}}$

For Radix-2 DIT FFT algorithm:-

$$X(K) = \sum_{n=\text{even}}$$

$\frac{N}{2}$ Point DFT

$$+ \sum_{n=\text{odd}}$$

$\frac{N}{2}$ Point DFT

For ex

$$N=6 \Rightarrow x(n) = \{x(0), x(1), x(2), x(3), x(4), x(5)\}$$

$$x_1(n) = \{x(0), x(2), x(4)\} \rightarrow \text{even numbered sequence}$$

$$x_2(n) = \{x(1), x(3), x(5)\} \rightarrow \text{odd numbered seq.}$$

$$X(K) = \sum_{n=0}^{\frac{N}{2}-1} x(2n) W_N^{2Kn} + \sum_{n=0}^{\frac{N}{2}-1} x(2n+1) W_N^{K(2n+1)}$$

even numbered
sequence

(sequence) نفرد
(even) الجزء ال

$$W_N^{2Kn} = \left(W_N^{Kn}\right)^2$$

$$W_N^{K(2n+1)} = W_N^{2Kn} \cdot W_N^K$$

$$X(K) = \sum_{n=0}^{\frac{N}{2}-1} x(2n) W_{N/2}^{Kn} + \sum_{n=0}^{\frac{N}{2}-1} x(2n+1) W_{N/2}^{Kn} \cdot W_N^K$$

$$= \sum_{n=0}^{\frac{N}{2}-1} x(2n) W_{N/2}^{Kn} + W_N^K \sum_{n=0}^{\frac{N}{2}-1} x(2n+1) W_{N/2}^{Kn}$$

assume: $F_1(n) = x(2n)$

$F_2(n) = x(2n+1)$

as (sequence) في $x(2n)$ $F_1(n)$

$$X(K) = \sum_{n=0}^{\frac{N}{2}-1} F_1(n) W_{N/2}^{Kn} + W_N^K \sum_{n=0}^{\frac{N}{2}-1} x(2n+1) W_{N/2}^{Kn}$$

$F_1(K) \leftarrow$ 11

$$X(K) = F_1(K) + W_N^K F_2(K) \rightarrow \boxed{1}$$

Where:

$F_1(K)$ is DFT for $F_2(n) = x(2n)$

$F_2(K)$ is DFT for $F_2(n) = x(2n+1)$

$F_1(K) \rightarrow \frac{N}{2}$ Point DFT

$F_2(K) \rightarrow \frac{N}{2}$ Point DFT

For eq(1) $\Rightarrow K = 0, 1, \dots, \frac{N}{2} - 1$

$\hookrightarrow X(K)$ has $\frac{N}{2}$ Point DFT

$$X(K + \frac{N}{2}) = F_1(K + \frac{N}{2}) + W_N^{K + \frac{N}{2}} F_2(K + \frac{N}{2})$$

$$X(K + \frac{N}{2}) = F_1(K) - W_N^K F_2(K)$$

$$K = 0, 1, \dots, \frac{N}{2} - 1$$

* For large values of N , this operation is repeated until we reach to 2-Point DFT Computation.

Ex: $x(n) = \{x(0), x(1)\} \Rightarrow N=2$

$$X(K) = \sum_{n=0}^{1} x(n) W_2^{Kn}$$

~~$x(0)W_2^0 + x(1)W_2^1$~~

$X(K) = x(0)W_2^0 + x(1)W_2^K$

~~$X(K) = x(0) + x(1)$~~

$$W_2^1 = e^{-j\frac{2\pi}{2}}$$

$$= e^{-j\pi} = -1$$

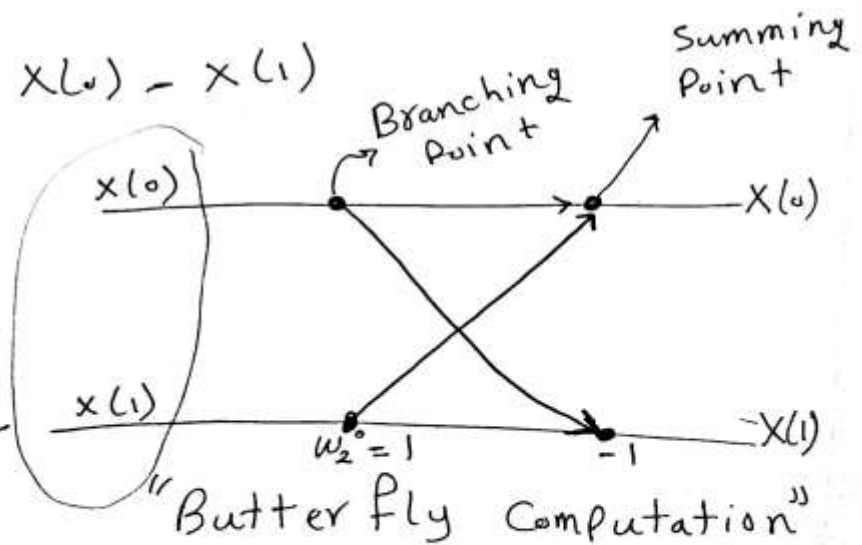
$K=0 \Rightarrow X(0) = x(0) + x(1)$

$K=1 \Rightarrow X(1) = x(0) - x(1)$

الشكل د. يوضح المعادلتين

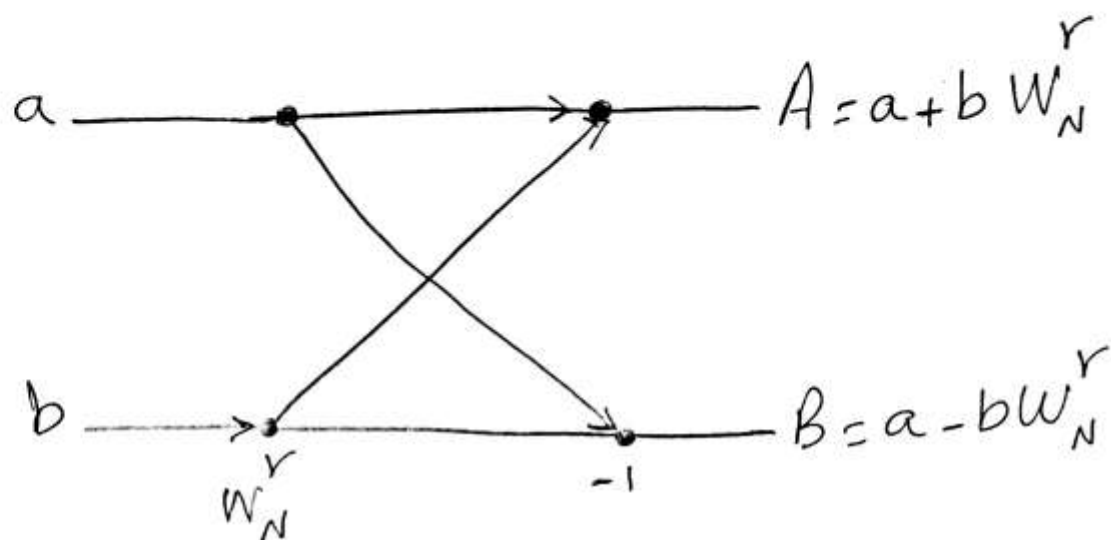
$X(0), X(1)$

2-Point Discrete
time Sequence



$x(0), x(1) \rightarrow$ 2-Point DFT.

In general



"General Form For Butterfly Computation"

Ex] For $N=4$

$$X(K) = F_1(K) + W_N^K F_2(K) \rightarrow \textcircled{1}$$

$$X\left(K + \frac{N}{2}\right) = F_1(K) - W_N^K F_2(K) \rightarrow \textcircled{2}$$

$$K = 0, 1, \dots, \frac{N}{2} - 1$$

Where:

$F_1(n) = X(2n) \rightarrow$ even numbered sequence

$F_2(n) = X(2n+1) \rightarrow$ odd "

$$X(K) = F_1(K) + W_4^K F_2(K) \rightarrow (1)$$

$$X(K+2) = F_1(K) - W_4^K F_2(K) \rightarrow (2)$$

$$K = 0, 1$$

eq (1)

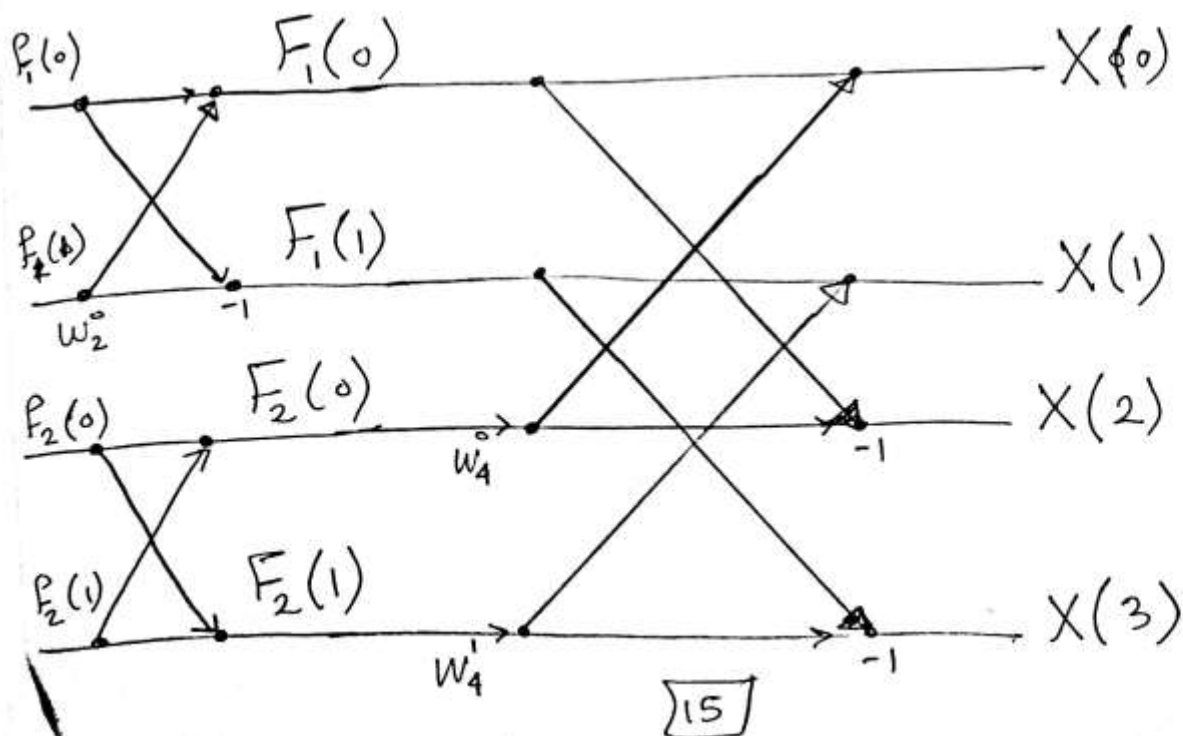
$$K=0 \Rightarrow \left\{ X(0) = F_1(0) + W_4^0 F_2(0) \right\} \sim 2\text{-Point DFT}$$

$$K=1 \Rightarrow \left\{ X(1) = F_1(1) + W_4^1 F_2(1) \right\}$$

in eqn 2

$$K=0 \Rightarrow \left\{ X(2) = F_1(0) - W_4^0 F_2(0) \right\} \sim 2\text{-Point DFT}$$

$$K=1 \Rightarrow \left\{ X(3) = F_1(1) - W_4^1 F_2(1) \right\}$$



في الرسم السابق كل علامة x تمثل (Butter fly)

$$P_1(n) = x(2n) \rightarrow n=0,1$$

$$P_1(n) = \{x(0), x(2)\}$$

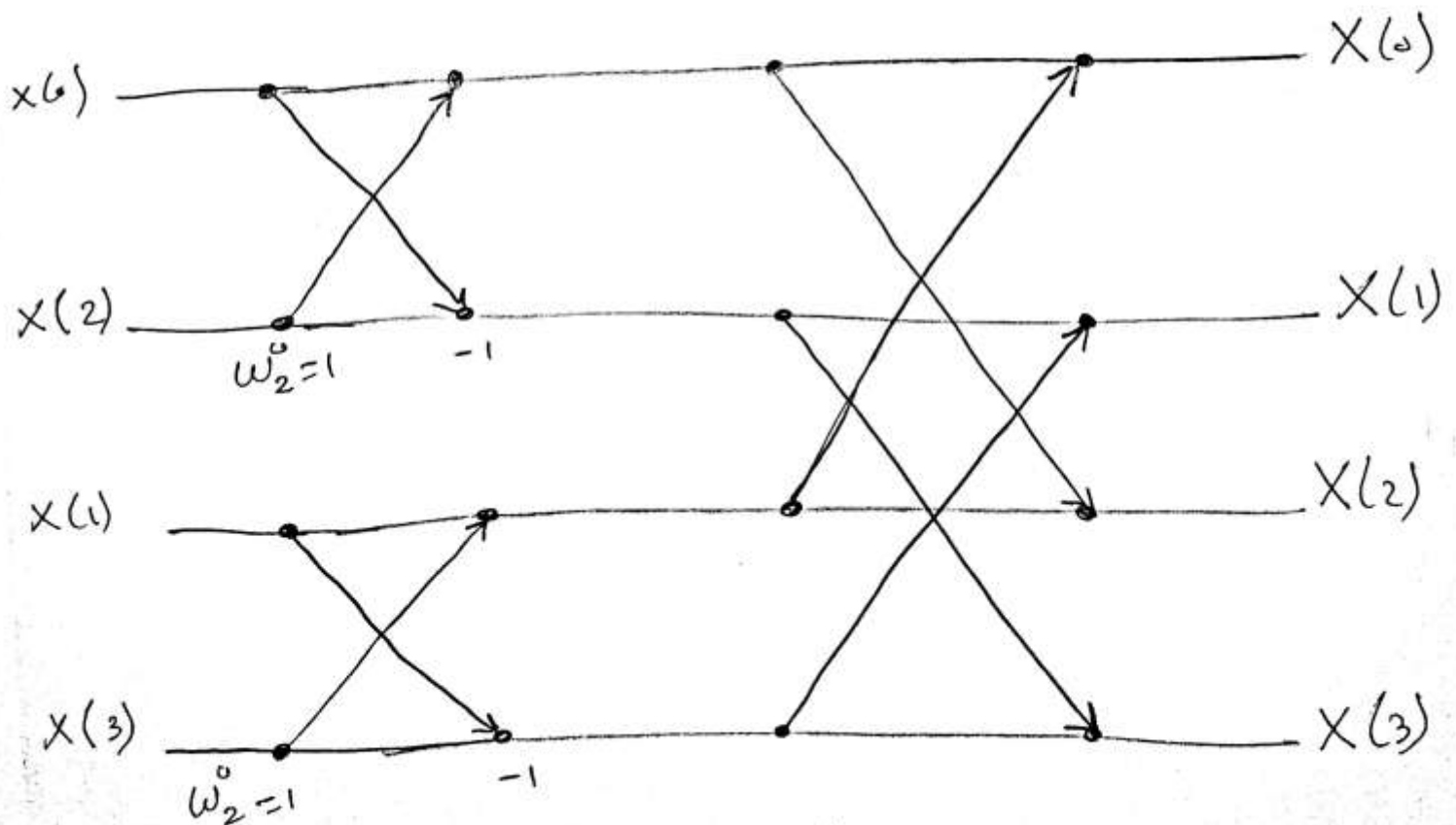
$$P_2(n) = x(2n+1), n=0,1$$

$$= \{x(1), x(3)\}$$

$$P_1(0) = x(0), P_1(1) = x(2)$$

$$P_2(0) = x(1), P_2(1) = x(3)$$

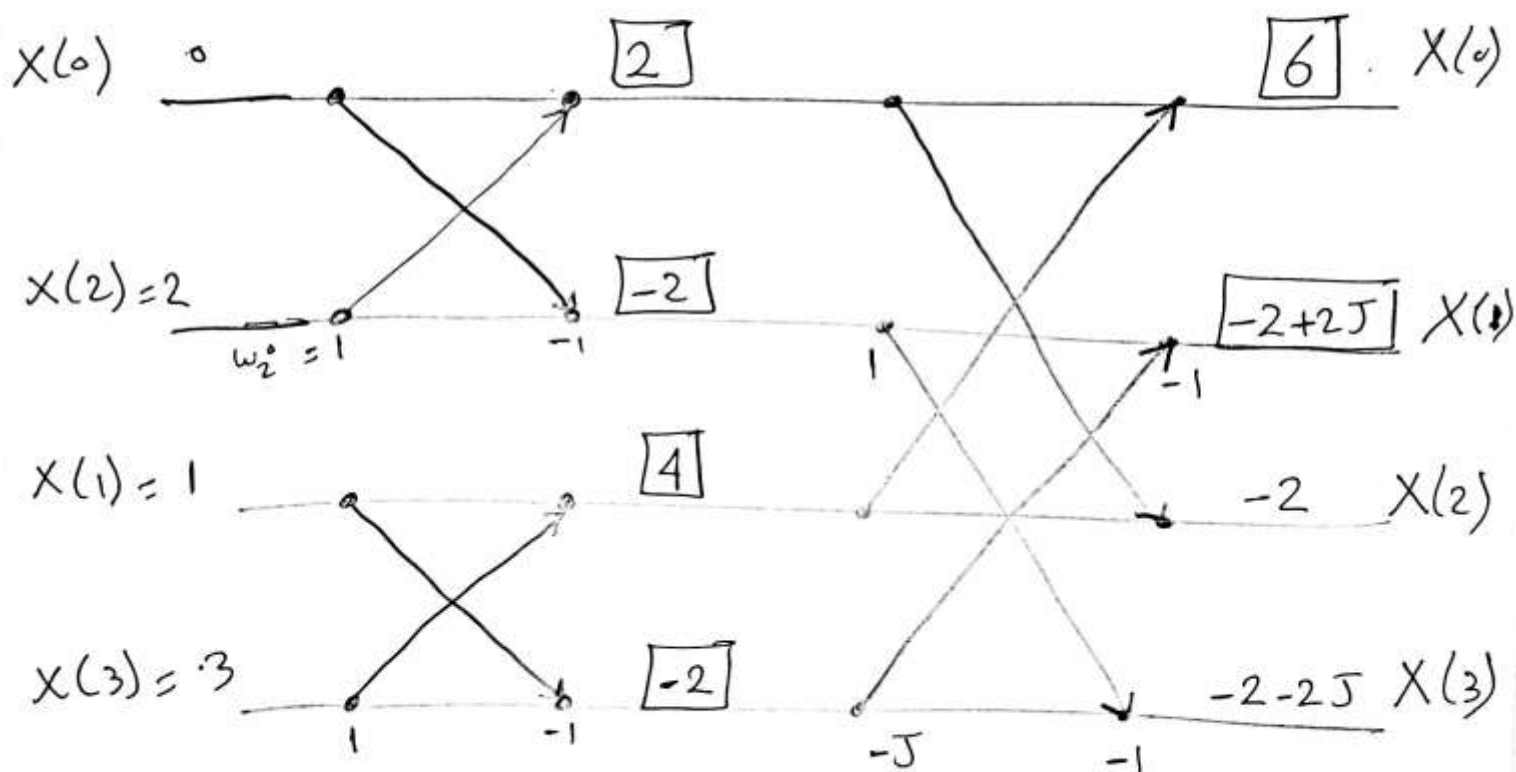
يمكن ترسيمها كـ



1/1

Ex Find radix-2 DIT FFT for

$$x(n) = \{0, 1, 2, 3\}$$



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